A GENERALIZATION OF NEHARI'S p - CRITERION FOR UNIVALENCE

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To Professor Peter Duren on his 70^{th} birthday

ABSTRACT. Nehari's most general univalence criterion $|Sf(z)| \leq 2p(|z|)$ requires the even solution of $u'' + pu = 0$ to have no zeros, and $(1 - x^2)^2 p(x)$ to be non-increasing on $[0, 1)$. In this paper, we show univalence under a weaker form of the second assumption, namely that pu^4 be non-increasing on $[0, 1)$.

1. Introduction

Let f be a locally univalent analytic function on the unit disk $\mathbb{D}$ and let

$$Sf = (f''/f')' - (1/2)(f''/f')^2$$

be its Schwarzian derivative. We owe to Nehari the important discovery of the connection between univalence, the Schwarzian derivative, and differential equations. Among his several results in this area, the one we will generalize states the following.

Theorem 1 (Nehari's p-criterion). *Let $p\colon (-1, 1) \to \mathbb{R}$ be a positive, even, continuous function satisfying the two conditions:*

(a) *the differential equation $u'' + pu = 0$ has no solution with more than one zero in $(-1, 1)$ other than the zero solution;*

(b) *$(1 - x^2)^2 p(x)$ is nonincreasing on $[0, 1)$.*

If

$$|Sf(z)| \leq 2p(|z|)\,, \quad z \in \mathbb{D}\,, \tag{1}$$

then f is univalent in $\mathbb{D}$. The constant 2 in (1) cannot be replaced by a larger number.

See [4] and [5]. The choices $p(x) = 1/(1 - x^2)^2$ and $p(x) = \pi^2/4$ give Nehari's original univalence criteria in [3]:

$$|Sf(z)| \leq \frac{2}{(1 - |z|^2)^2} \quad \text{and} \quad |Sf(z)| \leq \frac{\pi^2}{2}\,. \tag{2}$$

Nehari put condition (a) in the form: "The differential equation $u'' + pu = 0$ has a solution which does not vanish in $(-1, 1)$." This is easily seen to be equivalent to the condition as stated in the theorem. It is a little more natural for us to work with the formulation we have given. For example, we will make use of the solution of the initial value problem $u'' + pu = 0$, $u(0) = 1$, $u'(0) = 0$. This function is even and we can then immediately assert that it is also positive in $(-1, 1)$.

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The main purpose of this paper is to prove:

Theorem 2. *Let $p\colon [0,1) \to \mathbb{R}$ be a nonnegative, continuous function satisfying the two conditions:*

(a) *the solution of $u'' + pu = 0$ with $u(0) = 1$, $u'(0) = 0$ is positive on $[0,1)$;*
(b) *pu^4 is nonincreasing on $[0,1)$.*

If

$$|Sf(z)| \leq 2p(|z|)\,, \quad z \in \mathbb{D}\,, \tag{3}$$

then f is univalent in $\mathbb{D}$.

We refer to the class of functions p entering in Theorem 1 as *Nehari functions*. We shall establish that Theorem 2 includes Theorem 1 by showing that the class of functions p entering in Theorem 2 is broader than the class of Nehari functions. Condition (b) in Theorem 1 is an artefact of Nehari's proof and the same can fairly be said of condition (b) in Theorem 2 in the proof we give, though, in more than appearance, the respective ways the conditions are used are quite different. The proof of Theorem 2 will be given in Section 2. The relationship between the two theorems will be discussed in Section 4.

One may wonder what kinds of nonincreasing functions of the form pu^4 can appear in Theorem 2. Quite a few. We will show:

Theorem 3. *Let $\lambda\colon [0,1) \to \mathbb{R}$ be any positive, continuous, nonincreasing function. Then there exists a positive constant c and a positive, continuous function $p\colon [0,1) \to \mathbb{R}$ such that the initial value problem*

$$u'' + pu = 0\,, \quad u(0) = 1\,,\, u'(0) = 0\,,$$

has a positive solution on $[0,1)$ satisfying $pu^4 = c\lambda$.

The proof of this will be given in Section 3.

Nehari's p-criterion has found still another life in [1] applying to lifts of harmonic mappings to minimal surfaces. That is joint work with Peter Duren who has graciously both turned 70 and allowed us to independently offer this variant of the analytic case. We are pleased to dedicate this paper to him. We are also grateful to the referee for a very thorough reading of the paper and for many helpful comments.

2. Proof of Theorem 2

To prove Theorem 2 we appeal to the general univalence criterion in [6] specialized to the case at hand. The ingredients are these: Let $e^\sigma|dz|$ be a smooth conformal metric on $\mathbb{D}$, let $0 < \delta \leq \infty$ be the diameter of $\mathbb{D}$ with respect to this metric, and suppose any two points in $\mathbb{D}$ can be joined by a geodesic of length $< \delta$. According to [6], if

$$|Sf - 2(\sigma_{zz} - \sigma_z^2)| \leq 2\sigma_{z\bar{z}} + \frac{2\pi^2}{\delta^2}e^{2\sigma}\,, \tag{4}$$

then f is univalent in $\mathbb{D}$.

A complete metric has $\delta = \infty$ and the condition becomes

$$|Sf - 2(\sigma_{zz} - \sigma_z^2)| \leq 2\sigma_{z\bar{z}}\,. \tag{5}$$

The curvature of the metric is $K = -e^{-2\sigma}\Delta\sigma$ and this is where the term $\sigma_{z\bar{z}}$ comes from. The term $2(\sigma_{zz} - \sigma_z^2)$, or rather the difference between Sf and this term, comes from changing the metric conformally from the Euclidean metric $|dz|$ to $e^\sigma|dz|$ and from computing a generalized Schwarzian that depends on the metric.

One can recover Nehari's two original univalence criteria (2) by taking the metric $e^\sigma|dz|$ to be, respectively, the Poincaré metric (curvature -4, diameter ∞; complete), and the Euclidean metric (curvature 0, diameter 2; incomplete).

One can also recover Theorem 1 but this requires more work and was the main point of [2]. It was in following up on some of the phenomena in that paper that led us to Theorem 2.

Under the assumptions of Theorem 2, let u be the positive solution on $[0,1)$ of $u''+pu=0$, $u(0)=1$, $u'(0)=0$. We use u to define the radial conformal metric $u^{-2}(|z|)|dz|$ on $\mathbb{D}$. It follows from the fact that $u^{-2}(|z|)$ is radial and increasing that the metric disks about the origin are convex, and hence that any two points in $\mathbb{D}$ can be joined by a geodesic. The metric is complete if and only if

$$\int_0^1 u^{-2}(r)\,dr = \infty\,.$$

The proof of Theorem 2 consists of showing that the bound $|Sf(z)| \le 2p(|z|)$ in (3) implies the inequality (5) for $\sigma(z) = -2\log u(|z|)$. Since (5) is the stronger of the conditions (4) and (5) we can conclude that f is univalent even if the metric is incomplete, *i.e.*, if the integral above is finite. However, while completeness is not needed for the proof of Theorem 2, we will show that the metric may be assumed to be complete in deducing Theorem 1 from Theorem 2.

In terms of u and p the Gaussian curvature of the metric is

$$K(z) = -2u^4(r)(Au(r)+p(r))\,, \quad r=|z|\,, \tag{6}$$

where

$$Au(r) = \left(\frac{u'(r)}{u(r)}\right)^2 - \frac{1}{r}\frac{u'(r)}{u(r)}\,, \quad r=|z|\,. \tag{7}$$

The initial conditions on u imply that Au is continuous at 0, with $Au(0)=p(0)$. Moreover, because $u'(r)\le 0$ for $0\le r<1$ the curvature is negative.

A straightforward calculation now shows that shows that (5) becomes

$$\left|\zeta^2 Sf(z) + Au(|z|) - p(|z|)\right| \le Au(|z|) + p(|z|)\,, \quad \zeta = \frac{z}{|z|}\,, \tag{8}$$

and this is what we must establish. The role of the hypothesis that pu^4 is nonincreasing is to deduce the following inequality between Au and p.

Lemma 1. *Under the assumptions (a) and (b) of Theorem 2,*

$$Au(r) \ge p(r)\,, \quad 0\le r<1\,. \tag{9}$$

See [2] for a version of this lemma applying to Theorem 1.

Proof. For $r>0$ rewrite $Au(r)\ge p(r)$ as

$$ru^2(r)p(r) \le r(u'(r))^2 - u(r)u'(r)\,. \tag{10}$$

Suppose that $p(r)$ is C^1. It suffices to prove (10) for the derivatives because both sides vanish at $r = 0$. After some cancelations using $u'' + pu = 0$, this is equivalent to

$$\frac{p'(r)}{p(r)} \leq -4\frac{u'(r)}{u(r)}, \quad 0 \leq r < 1,$$

which holds because pu^4 is non-increasing. Thus (10) holds, and since $Au(0) = p(0)$, as noted above, it follows that $Au(r) \geq p(r)$ for $r \geq 0$. In the general case, when p is just continuous, simply approximate p uniformly on compact sets by smooth functions. $\square$

To finish the proof of Theorem 2, suppose that $|Sf(z)| \leq 2p(|z|)$. Then, because of Lemma 1, we have

$$\begin{aligned}\left|\zeta^2 Sf(z) + Au(|z|) - p(|z|)\right| &\leq |Sf(z)| + Au(|z|) - p(|z|) \\ &\leq 2p(|z|) + Au(|z|) - p(|z|) = Au(|z|) + p(|z|).\end{aligned}$$

Hence (8) holds, so that f is univalent in $\mathbb{D}$.

3. Proof of Theorem 3

To prove Theorem 3, for any positive, nonincreasing, continuous function λ on $[0, 1)$ we must produce a positive, continuous function p so that,

$$pu^4 = c\lambda,$$

for some $c > 0$, where u is the positive solution on $[0, 1)$ to the initial value problem

$$u'' + pu = 0, \quad u(0) = 1, \, u'(0) = 0.$$

Our approach is to show that there is an $\alpha > 0$ so that the solution to the initial value problem

$$w'' + \lambda w^{-3} = 0, \quad w(0) = \alpha, \quad w'(0) = 0 \tag{11}$$

is positive on $[0, 1)$. If so, we set $p = -w''/w$, and then the function $u = (1/\alpha)w$ satisfies $u'' + pu = 0$, $u(0) = 1$, $u'(0) = 0$ and $pu^4 = c\lambda$, where $c = 1/\alpha^4$.

To accomplish this we formulate the following comparison lemma.

Lemma 2. *Let λ, ρ be positive, continuous functions on $[0, 1)$ with $\lambda \leq \rho$. Let w and v be, respectively, the solutions of*

$$\begin{aligned} w'' + \lambda w^{-3} = 0\,, \quad w(0) = \alpha\,,\ w'(0) = 0\,, \\ v'' + \rho v^{-3} = 0\,, \quad v(0) = \beta\,,\ v'(0) = 0\,, \end{aligned}$$

where $\alpha \geq \beta > 0$. Then $w \geq v$ up to the first zero of v.

Proof. First suppose that $\alpha > \beta$ with strict inequality. Let $h = w'v - wv'$. Then $h(0) = 0$, and as long as $w > v > 0$ we have

$$h' = w''v - wv'' = \rho wv^{-3} - \lambda vw^{-3} \geq \lambda(wv^{-3} - vw^{-3}) > 0.$$

But $h' > 0$ on an interval $(0, a)$ implies that $h > 0$ and hence that $w > v$ on $(0, a)$. The inequality $w > v$ thus persists till the first zero a of v. Therefore the lemma obtains in this case since initially $w > v > 0$ near the origin. The implication $\alpha \geq \beta > 0 \implies w \geq v$ up to the first zero of v follows by a limiting argument. $\square$

This applies to the proof of Theorem 3 as follows. Since λ is nonincreasing,

$$\lambda(x) \le \lambda(0) = \lambda_0 \,,$$

and we conclude from the lemma that the solution u of (11) cannot vanish before the solution of

$$v'' + \lambda_0 v^{-3} = 0 \,, \quad v(0) = \alpha \,,\; v'(0) = 0 \,.$$

This can be determined explicitly and is given by

$$v(x) = \frac{1}{\alpha}\sqrt{\alpha^4 - \lambda_0 x^2} \,.$$

The positive zero is $x = \alpha^2/\sqrt{\lambda_0}$ and so to get a positive solution v (and hence w) on $[0,1)$ we take

$$\alpha \ge \lambda_0^{1/4} \,.$$

4. The relationship between Theorem 1 and Theorem 2

We now want to explain why Theorem 2 is more general than Theorem 1, first, specifically under what circumstances the condition

$$(1-x^2)^2 p(x) \quad \text{is nonincreasing}$$

in Theorem 1 implies the condition

$$pu^4 \quad \text{is nonincreasing}$$

in Theorem 2, and then how the latter is more encompassing.

The analysis of the first point depends on an important scaling phenomenon of Nehari functions. It was shown in [2] that for any Nehari function p there exists a (maximal) value $t_0 \ge 1$ such that $t_0 p$ remains a Nehari function, the issue being whether scaling p to tp, $1 \le t \le t_0$ maintains the property that the equation $u'' + tpu = 0$ has no nontrivial solutions that vanish more than once in $(-1,1)$. The result can be described neatly in terms of the extremal function associated with the criterion. For $t \ge 1$ let u be the solution of $u'' + tpu = 0$ with $u(0) = 1$, $u'(0) = 0$, and define

$$\Phi_t(x) = \int_0^x u^{-2}(s)\, ds \,.$$

Then $\Phi_t(0) = 0$, $\Phi_t'(0) = 1$, $\Phi_t''(0) = 0$, $S\Phi_t = 2tp$, and as long as tp remains a Nehari function Φ_t is defined on $(-1,1)$. A result in [2] is that $t_0 > 1$ if and only if $\Phi_t(1) < \infty$ for $1 \le t < t_0$, and that for $t_0 p$ we have $\Phi_{t_0}(1) = \infty$. Geometrically, this means that the metric $\Phi_{t_0}'(|z|)|dz|$ is complete, and this is used in a number of constructions in [2].

Suppose now that the hypotheses of Theorem 1 hold. In any use of the theorem, since the inequality $|Sf(z)| \le 2p(|z|)$ trivially implies that $|Sf(z)| \le 2tp(|z|)$ for $t > 1$, we can replace p by $t_0 p$ (if necessary) and assume without loss of generality that

$$\int_0^1 u^{-2}(s)\, ds = \infty \tag{12}$$

for the positive function u with

$$u'' + pu = 0 \,, \quad u(0) = 1 \,, u'(0) = 0 \,. \tag{13}$$

Under the assumption (12) we will now show that pu^4 is nondecreasing on $[0,1)$, and it is in this sense that Theorem 1 is subsumed by Theorem 2. Recall from the proof of Lemma 1 that to say pu^4 is nonincreasing on $[0,1)$ is the same as

$$\frac{p'(x)}{p(x)} \leq -4\frac{u'(x)}{u(x)}\,, \quad 0 \leq x < 1\,. \tag{14}$$

This is the key inequality and we will work with several equivalent forms of it in what follows.

Let

$$q(x) = (1-x^2)^2 p(x)\,,$$

and define

$$w(x) = \frac{u(x)}{\sqrt{1-x^2}}\,.$$

Then (13) leads to

$$(1-x^2)((1-x^2)w'(x))' = (1-q(x))w(x)\,, \quad w(0)=1\,,\, w'(0)=0\,. \tag{15}$$

We can write this more compactly by introducing the change of variables $x(s) = \tanh s$, $0 \leq s < \infty$, for which $x'(s) = 1 - x^2(s)$. With

$$\varphi(s) = w(x(s)) \quad \text{and} \quad \nu(s) = q(x(s))$$

equation (15) is

$$\varphi''(s) = (1-\nu(s))\varphi(s)\,, \quad \varphi(0)=1\,,\, \varphi'(0)=0\,. \tag{16}$$

The function $\nu(s)$ is nonincreasing because $q(x)$ is nonincreasing, a hypothesis in Theorem 1, and it was shown in [2] that when (12) holds the solution φ of (16) is strictly decreasing for $s > 0$. Therefore w is decreasing on $[0,1)$. But now a simple calculation shows that the inequality (14) is equivalent to

$$\frac{q'(x)}{q(x)} \leq -4\frac{w'(x)}{w(x)}\,, \tag{17}$$

and this will be true because $q' \leq 0 \leq -w'$. To reiterate, in Theorem 1 when (12) is true, which is no loss of generality, it follows that pu^4 is nonincreasing.

Now let us show that just requiring pu^4 to be nonincreasing – the second hypothesis in Theorem 2 – allows for a larger class of functions p than the Nehari functions. To construct examples it is easier to work with $\nu(s)$ and $\varphi(s)$ and to write the inequality (17) as

$$\frac{\nu'(s)}{\nu(s)} \leq -4\frac{\varphi'(s)}{\varphi(s)}\,. \tag{18}$$

We seek p such that:

(i) The corresponding function $\nu(s)$ fails to be nonincreasing, but still
(ii) The solution φ of (16) is positive, and
(iii) pu^4 is nonincreasing, *i.e.*, (18) holds, and, moreover, (12) also holds.

It is φ that we will find.

Examples of interest will have $\nu(s)$, and therefore $q(x) = (1-x^2)^2 p(x)$, oscillate infinitely often around the value 1. In effect, if $\nu(s) < 1$ for all $s > s_0$, then $p(x) < 1/(1-x^2)^2$ for $x > x_0$, a case in some sense already known, while, on the other hand, ν cannot remain larger than 1 too long without forcing φ to vanish. Oscillations of ν around 1 force φ to change concavity infinitely often, and this is the clue.

We can be explicit. We begin with a function of the form

$$\psi(s) = e^{-as}(1 + b\sin s)\,.$$

Tedious but simple calculations show that there is a crescent-shaped region near the origin in the first quadrant of the ab-plane consisting of points (a, b) such that ψ is positive, decreasing for $s > 0$, changes concavity infinitely often, and satisfies the corresponding version of (18), namely

$$\frac{\mu'(s)}{\mu(s)} \leq -4\frac{\psi'(s)}{\psi(s)}\,, \tag{19}$$

when $\mu = 1 - (\psi''/\psi)$. Furthermore, since $\psi(s) \to 0$ as $s \to \infty$, it follows that $w(x)$ (defined using this ψ) also tends to 0 as $x \to 1$, whence (12) is valid.

This almost suffices, but we require a small modification to satisfy the initial conditions in (16) while not destroying the other properties. To achieve this, take a point s_1 where $\mu(s_1) > 1$ and let

$$\mu_1(s) = \begin{cases} \mu(s_1)\,, & s \leq s_1 \\ \mu(s)\,, & s_1 \leq s\,. \end{cases}$$

Now let ψ_1 be the solution of

$$\psi_1''(s) = (1 - \mu_1(s))\psi_1(s)\,, \quad \psi_1(s_1) = \psi(s_1)\,, \psi_1'(s_1) = \psi'(s_1)\,.$$

That is, we keep $\psi(s)$ for $s_1 \leq s$ and smoothly attach a trigonometric function for $s \leq s_1$.

Let $s_0 < s_1$ be the point where the first maximum of $\psi_1(s)$ occurs. Since $\psi_1'(s_1) = \psi'(s_1) < 0$ we know that $\psi_1(s)$ decreases from s_0 to s_1, and hence this modification of ψ does not destroy the inequality (19) for $s_0 \leq s \leq s_1$ because the left hand side vanishes while the right hand side is positive. Finally, by defining the functions

$$\varphi(s) = \frac{1}{\psi_1(s_0)}\psi_1(s + s_0)\,, \quad \nu(s) = \mu(s + s_0)\,,$$

we shift s_0 to the origin and scale. The functions φ and ν satisfy

$$\varphi'(s) = (1 - \nu(s))\varphi(s)\,, \quad \varphi(0) = 1\,, \varphi'(0) = 0\,,$$

as well as (18) and (12). This completes the example.

References

1. M. Chuaqui, P. Duren, and B. Osgood, *Univalence criteria for lifts of harmonic mappings to minimal surfaces*, (to appear).
2. M. Chuaqui and B. Osgood, *Finding complete conformal metrics to extend conformal mappings*, Indiana Univ. Math. J. **47** (1998), 1273–1291.
3. Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. **55** (1949), 545–551.
4. ______, *Some criteria of univalence*, Proc. Amer. Math. Soc. **5** (1954), 700–704.
5. ______, *Univalence criteria depending on the Schwarzian derivative*, Illinois J. Math. **23** (1979), 345–351.
6. B. Osgood and D. Stowe, *A generalization of Nehari's univalence criterion*, Comm. Math. Helv. **65** (1990), 234–242.

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